Cornell PHYS 2218, Problem Set #1 Solutions

Pius Lee (academics@plee.app)

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1.1. For the mass and spring discussed (1.1)-(1.8), suppose that the system is hung vertically in the earth's gravitational field, with the top of the spring held fixed. Show that the frequency for vertical oscillations is given by (1.5). Explain why gravity has no effect on the angular frequency.

For a vertical mass-spring system in a gravitational field, the spring stretches by $x_0 = \frac{mg}{k}$ to balance gravity. Displacing the mass by x from this new equilibrium position, the total restoring force is:

$$F_{\text{total}} = -k(x_0 + x) + mg = -kx$$

This shows the restoring force is the same as in a horizontal system, and the equation of motion remains:

$$m\ddot{x} = -kx$$

Thus, the angular frequency is

$$\omega = \sqrt{\frac{k}{m}}$$

Gravity only shifts the equilibrium position but doesn't affect the restoring force, so the frequency is the same as in the absence of gravity.

1.2.a. I'm not typing this out.

1.3. Suppose that z_1 and z_1, two points in the complex plane, correspond to two vertices of an equilateral triangle. Show that the third vertex is given by $z_3 = -\zeta z_1 - \zeta^2 z_2$, with $\zeta^3 = 1$ a cube root of unity.

The cube root of unity aligned w.r.t. $\zeta^3 = 1$ contains the following scalars:

$$\{\zeta^1, \zeta^2, \zeta^3\} = \left\{\exp\left(\frac{2k\pi i}{3}\right)\right\} = \left\{\frac{-1 + i\sqrt{3}}{2},\ \frac{-1 - i\sqrt{3}}{2},\ 1\right\}$$

For any non-zero z_k, $l = |z_1| = |z_2| = |z_3|$, since all points on an equilateral triangle are equidistant. Therefore,

$$l = \zeta l - \zeta^2 l \implies 1 = -\zeta - \zeta^2 = -\frac{-1 + i\sqrt{3}}{2} - \frac{-1 - i\sqrt{3}}{2}$$

$$\boxed{1 = 1} \qquad \blacksquare$$

Info for 1.4. A block of mass M slides without friction between two springs of spring constant K and $2K$, as shown (omitted). The block is constrained to move only left and right on the paper, so the system has only one degree of freedom.

1.4.a. Calculate the oscillation angular frequency.

The effective spring coefficient is $K_{\text{eff}} = K + 2K$ because the springs are in series. The frequency is found with the equation,

$$\omega = \sqrt{\frac{K_{\text{eff}}}{m}} \implies \sqrt{\frac{3K}{m}}$$

1.4.b. If the velocity of the block when it is at its equilibrium position is v, calculate the amplitude of the oscillation.

$$\sum E = \frac{1}{2}Mv^2 = \frac{1}{2}K_{\text{eff}}A^2 = \frac{3K}{2}A^2$$

Solving for A yields:

$$\frac{1}{2}Mv^2 = \frac{3K}{2}A^2 \implies \boxed{A = \sqrt{\frac{Mv^2}{3K}}}$$

Info for 1.5. A particle of mass m moves on the x axis with potential energy

$$V(x) = \frac{E_0}{a^4}\left(x^4 + 4ax^3 - 8a^2x^2\right).$$

1.5.a. Find the positions at which the particle is in stable equilibrium.

Stable equilibrium is defined as a potential well. So, you find solutions for $0 = \frac{dV}{dt}$ and ensure $\frac{d^2V}{dt^2} > 0$. This turns out to be

$$\boxed{x = \{-4a,\ a\}}$$

1.5.b. Find the angular frequency of small oscillations about each equilibrium position. What do you mean by small oscillations? Be quantitative and give a separate answer for each point of stable equilibrium.

By *small oscillations*, I think they mean oscillations at the limit of each solution. We cannot have any oscillations at stable equilibrium because the system is stationary. Since $k = m\omega^2$ and $K_{\text{eff}} = \frac{d^2V}{dt^2}$ at equilibrium, we yield

$$\boxed{\omega(x) = \{-4a,\ a\} \mapsto \left\{\sqrt{\frac{80E_0}{ma^2}}, \sqrt{\frac{20E_0}{ma^2}}\right\}}$$

Info for 1.6. A puck of mass m slides without friction on a flat table.

There is a small hole in the table, and a massless inextensible string passing through the hole connects the puck to a block, also of mass m, that hangs below the table.

1.6(a). At $t = 0$ the block is instantaneously at rest, and the puck is given a push that sets it into instantaneously circular motion about the hole. For what value ω_{eq} of the angular frequency ω of this motion does the system remain in equilibrium, with the puck revolving at a fixed radius.

Centripetal motion (and force) provides us:

$$F_c = \frac{mv^2}{r} = mr\omega^2$$

because $\omega = \frac{v}{r}$. We also know $T = mg$. Since the system is at equilibrium, $F_{\text{net}} = 0$ so:

$$F_{\text{net}} = mr_{eq}\omega_{eq}^2 - mg.$$

Simple algebra yields us

$$g = r_{eq}\omega_{eq}^2 \implies \omega_{eq} = \sqrt{\frac{g}{r_{eq}}}.$$

1.6(b). Assume the puck is revolving at $\omega = \omega_{eq}$, and the block is hanging at a fixed height. At $t = t_1$ an external force gives the block a quick, small downward tug. Find the frequency of the resulting small oscillations.

At equilibrium, we know that:

$$T_{eq} = m\vec{g} = mr_{eq}\omega_{eq}^2 \implies \omega_{eq} = \sqrt{\frac{g}{r_{eq}}}$$

Now, let us consider a small downward displacement, δr. This tug modifies the radius of the circular motion, so

$$r = r_{eq} - \delta r.$$

I justify using a *negative* symbol, because it is a tug downwards. To restore this system to equilibrium, the weights must provide a restorative force (since that's the only force acting upon the system). Thus,

$$F_c = T = m(r_{eq} - \delta r)\omega^2$$

For the system to maintain angular momentum, $r^2\omega$ must remain constant under small osculations. At equilibrium, angular momentum is provided as:

$$L = mr_{eq}^2\omega_{eq}$$

Introducing a perturbation would provide an angular momentum of

$$L = m(r_{eq} - \delta r)^2\omega.$$

Since angular momentum is conserved,

$$\cancel{m} r_{eq}^2 \omega_{eq} = \cancel{m}(r_{eq} - \delta r)^2 \omega.$$

For small δr, $(r_{eq} - \delta r)^2 = r_{eq}^2 - 2r_{eq}\delta r + \cancel{(\delta r)^2}$. Plugging this in,

$$r_{eq}^2 \omega_{eq} = (r_{eq}^2 - 2r_{eq}\delta r)\omega.$$

Solving for ω yields

$$\omega \approx \omega_{eq}\left(1 + \frac{\delta r}{r_{eq}}\right).$$

Let's plug this in to our tension function.

$$\begin{aligned}
T(r) = mr\omega^2 &= m(r_{eq} - \delta r)\left(\omega_{eq}\left(1 + \frac{\delta r}{r_{eq}}\right)\right)^2 \\
&= m\omega_{eq}^2 (r_{eq} - \delta r)\left(1 + \frac{2\delta r}{r_{eq}} + \cancel{\frac{(\delta r)^2}{r_{eq}^2}}\right) \\
&= m\omega_{eq}^2 \left(r_{eq} - \delta r - \cancel{\frac{2\delta r^2}{r_{eq}^2}} + 2\delta r\right) \\
&= m\omega_{eq}^2 (r_{eq} + \delta r)
\end{aligned}$$

Since we know $m\omega_{eq}^2 r_{eq} = mg$ as shown in 1.6(a), we find that

$$T(r) = mg + m\omega_{eq}^2(\delta r).$$

The mg term cancels out when applying $T(r)$ onto F_{net}.

$$F_{\text{net}} = T(r) - mg = \cancel{mg} + m\omega_{eq}^2(\delta r) - \cancel{mg} = m\omega_{eq}^2(\delta r).$$

Since the "tug" acts as a change in position, this system is in acceleration.

$$F_{\text{net}} = m(\delta \ddot{r}) = m\omega_{eq}^2(\delta r)$$

Thus, we obtain the equation for a **simple harmonic oscillator**.

$$0 = m(\delta \ddot{r}) - m\omega_{eq}^2(\delta r)$$

Since we know $k = m\omega^2$, and $m\omega_{eq}^2$ serves as the k term,

$$m\omega^2 = m\omega_{eq}^2 \quad \Longrightarrow \quad \omega = \omega_{eq}$$

The question asks us to find the **frequency** of the perturbation of r (as opposed to the angular frequency). Therefore, we need to convert our answer to a linear expression with the factor $\omega = 2\pi f$. In conclusion,

$$\boxed{f = \frac{1}{2\pi}\omega_{eq}} \qquad \blacksquare$$